

Cayley-Type Conditions for Billiards within k Quadrics in \mathbb{R}^d

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Abstract

The notions of reflection from outside, reflection from inside and signature of a billiard trajectory within a quadric are introduced. Cayley-type conditions for periodical trajectories for the billiard in the region bounded by k quadrics in \mathbb{R}^d and for the billiard ordered game within k ellipsoids in \mathbb{R}^d are derived. In a limit, the condition describing periodic trajectories of billiard systems on a quadric in \mathbb{R}^d is obtained.

1 Introduction

We study periodic trajectories of the following well-known integrable mechanical system: motion of a free particle within an ellipsoid in the Euclidean space of any dimension d . On the boundary, the particle respects the billiard law. To be more precise, let us mention some basic notions.

Let (Q, g) be a d -dimensional Riemannian manifold and let $D \subset Q$ be a domain with a piecewise smooth boundary \mathcal{B} . Let $\pi : T^*Q \rightarrow Q$ be a natural projection and let g^{-1} be the contravariant metric on the cotangent bundle.

Consider the *reflection mapping* $r : \pi^{-1}\mathcal{B} \rightarrow \pi^{-1}\mathcal{B}$, $p_- \mapsto p_+$, which associates the covector $p_+ \in T_x^*Q$, $x \in \mathcal{B}$ to a covector $p_- \in T_x^*Q$ such that the following conditions hold: $|p_+| = |p_-|$, $p_+ - p_- \perp \mathcal{B}$.

A *billiard* in D is a dynamical system with the phase space $M = T^*D$ whose trajectories are geodesics given by the Hamiltonian $H(p, x) = \frac{1}{2}g_x^{-1}(p, p)$, reflected at points $x \in \mathcal{B}$ according to the billiard law: $r(p_-) = p_+$. Here p_- and p_+ denote the momenta before and after the reflection.

Integrability of the billiard system within quadrics is related to classical geometrical properties: the Chasles, Poncelet and Cayley theorems. From the Chasles theorem [1] every line in \mathbb{R}^d is tangent to $d-1$ quadrics confocal to the

boundary; all segments of one billiard trajectory are tangent to the same $d - 1$ quadrics [2]. We refer to these $d - 1$ quadrics as caustics of the given trajectory.

From now on, we consider a billiard with a boundary which consists of the union of k confocal quadrics in \mathbb{R}^d . According to the generalized Poncelet theorem [3], with k, d arbitrary, *there exists a closed trajectory with $d - 1$ given confocal caustics if and only if infinitely many such trajectories exist, and all of them have the same period*. The periodicity of a billiard trajectory depends on its caustic surfaces. An important question is to find an analytical connection between them and the corresponding period.

In [4], Cayley found the analytical condition for caustic conics in the Euclidean plane case ($d = 2$) with $k = 1$ conic as a boundary. The classical and algebro-geometric proofs of Cayley's theorem can be found in Lebesgue's book [5] and the paper [6], respectively. Moreover, in [5] the complete Poncelet theorem for billiard systems in a plane ($d = 2$) within k conics, with k arbitrary, was proved (see also [7]).

The generalisation of Cayley's condition for $k = 1$ is established by the authors for any dimension d [8], by use of the Veselov-Moser discrete $L - A$ pair [9].

The main goal of this paper is to give Cayley-type conditions describing periodic trajectories of the billiard in the region bounded by k confocal quadrics in \mathbb{R}^d and of the billiard ordered game within k ellipsoids in \mathbb{R}^d , for k, d arbitrary. The importance of these questions was underlined several times by experts; let us mention Arnol'd (see [1, 10]), for example, in connection with applications in laser technology. In a limit case, we derive analytic conditions for periodic billiard trajectories on a quadric in \mathbb{R}^d bounded by any finite number of quadrics, solving in this way a problem explicitly posed by Abenda and Fedorov [11].

2 Planar case: $d = 2, k$ arbitrary

The derivation of Cayley-type conditions for the billiard in a plane within k conics can be done following Lebesgue. In [5], he considered polygons inscribed in a conic Γ , whose sides are tangent to $\Gamma_1, \dots, \Gamma_k$, where $\Gamma, \Gamma_1, \dots, \Gamma_k$ all belong to a pencil of conics. In the dual plane, such polygons correspond to billiard trajectories having caustic Γ^* with bounces on $\Gamma_1^*, \dots, \Gamma_k^*$. The main object of Lebesgue's analysis in [5] was the cubic Cayley curve, which parametrizes contact points of tangents drawn from a given point to all conics of the pencil.

We summarize Lebesgue's results as follows. Let C and Γ be conics of a pencil \mathcal{F} and $\Delta(x)$ be the discriminant of the conic $C + x\Gamma = 0$. If $\lambda_1, \dots, \lambda_k$ denote parameters corresponding to $\Gamma_1, \dots, \Gamma_k$, respectively, then the existence of the Poncelet polygon is equivalent to

$$\det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^p & \sqrt{\Delta(\lambda_1)} & \lambda_1 \sqrt{\Delta(\lambda_1)} & \dots & \lambda_1^{p-2} \sqrt{\Delta(\lambda_1)} \\ \dots & \dots & \dots & & \dots & \dots & \dots & & \dots \\ 1 & \lambda_k & \lambda_k^2 & \dots & \lambda_k^p & \sqrt{\Delta(\lambda_k)} & \lambda_k \sqrt{\Delta(\lambda_k)} & \dots & \lambda_k^{p-2} \sqrt{\Delta(\lambda_k)} \end{pmatrix} = 0$$

for $k = 2p$

$$\det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^p & \sqrt{\Delta(\lambda_1)} & \lambda_1\sqrt{\Delta(\lambda_1)} & \dots & \lambda_1^{p-1}\sqrt{\Delta(\lambda_1)} \\ \dots & \dots \\ 1 & \lambda_k & \lambda_k^2 & \dots & \lambda_k^p & \sqrt{\Delta(\lambda_k)} & \lambda_k\sqrt{\Delta(\lambda_k)} & \dots & \lambda_k^{p-1}\sqrt{\Delta(\lambda_k)} \end{pmatrix} = 0$$

for $k = 2p + 1$.

The case with two ellipses, when the billiard trajectory is placed between them and the particle bounces from one to the other of them alternately, is of special interest.

Corollary 1 *The condition for existence of a $2m$ -periodic billiard trajectory which bounces exactly m times to the ellipse $\Gamma_1^* = C^*$ and m times to $\Gamma_2^* = (C + \gamma\Gamma)^*$, having Γ^* for the caustic, is*

$$\det \begin{pmatrix} f_0(0) & f_1(0) & \dots & f_{2m-1}(0) \\ f'_0(0) & f'_1(0) & \dots & f'_{2m-1}(0) \\ \dots & \dots & \dots & \dots \\ f_0^{(m-1)}(0) & f_1^{(m-1)}(0) & \dots & f_{2m-1}^{(m-1)}(0) \\ f_0(\gamma) & f_1(\gamma) & \dots & f_{2m-1}(\gamma) \\ f'_0(\gamma) & f'_1(\gamma) & \dots & f'_{2m-1}(\gamma) \\ \dots & \dots & \dots & \dots \\ f_0^{(m-1)}(\gamma) & f_1^{(m-1)}(\gamma) & \dots & f_{2m-1}^{(m-1)}(\gamma) \end{pmatrix} = 0$$

where $f_j = x^j$ ($0 \leq j \leq m$), $f_{m+i} = x^{i-1}\sqrt{\Delta(x)}$ ($1 \leq i \leq m-1$).

We consider a simple example with four bounces on each of the two conics.

Example 1 *The condition on a billiard trajectory placed between ellipses Γ_1^* and Γ_2^* , to be closed after four alternating bounces to each of them is*

$$\det X = 0$$

where the elements of the 3×3 matrix X are

$$\begin{aligned} X_{11} &= -4B_0 + B_1\gamma + 4C_0 + 3C_1\gamma + 2C_2\gamma^2 + C_3\gamma^3 \\ X_{12} &= -3B_0 + B_1\gamma + 3C_0 + 2C_1\gamma + C_2\gamma^2 \\ X_{13} &= -2B_0 + B_1\gamma + 2C_0 + C_1\gamma \\ X_{21} &= -6B_0 + B_2\gamma^2 + 6C_0 + 6C_1\gamma + 4C_2\gamma^2 + 3C_3\gamma^3 \\ X_{22} &= -6B_0 + B_1\gamma + B_2\gamma^2 + 6C_0 + 4C_1\gamma + 3C_2\gamma^2 \\ X_{23} &= -5B_0 + 2B_1\gamma + B_2\gamma^2 + 5C_0 + 3C_1\gamma \\ X_{31} &= -4B_0 + B_3\gamma^3 + 4C_0 + 4C_1\gamma + 4C_2\gamma^2 + 3C_3\gamma^3 \\ X_{32} &= -4B_0 + B_2\gamma^2 + B_3\gamma^3 + 4C_0 + 4C_1\gamma + 3C_2\gamma^2 \\ X_{33} &= -4B_0 + B_1\gamma + B_2\gamma^2 + B_3\gamma^3 + 4C_0 + 3C_1\gamma \end{aligned}$$

with C_i, B_i being coefficients in the Taylor expansions around $x = 0$ and $x = \gamma$, respectively

$$\begin{aligned}\sqrt{\Delta(x)} &= C_0 + C_1x + C_2x^2 + \dots \\ \sqrt{\Delta(x)} &= B_0 + B_1(x - \gamma) + B_2(x - \gamma)^2 + \dots\end{aligned}$$

3 Periodic billiard trajectories inside k confocal quadrics in \mathbb{R}^d

The complete Poncelet theorem (CPT) was generalized to the case $d = 3$ by Darboux in [12] in 1870. Higher-dimensional generalizations of CPT were obtained quite recently in [3]. The main result of the present paper is the Cayley-type condition for generalized CPT for $d \geq 3$, although obtained results can be applied immediately in the case $d = 2$.

Consider an ellipsoid in \mathbb{R}^d

$$\frac{x_1^2}{a_1} + \dots + \frac{x_d^2}{a_d} = 1 \quad a_1 > \dots > a_d > 0$$

and a related system of Jacobian elliptic coordinates $(\lambda_1, \dots, \lambda_d)$ ordered by the condition

$$\lambda_1 > \lambda_2 > \dots > \lambda_d.$$

Any quadric from the corresponding confocal family is given by

$$Q_\lambda : \frac{x_1^2}{a_1 - \lambda} + \dots + \frac{x_d^2}{a_d - \lambda} = 1. \quad (1)$$

Lemma 1 Suppose a line ℓ is tangent to quadrics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ from the family (1). Then Jacobian coordinates $(\lambda_1, \dots, \lambda_d)$ of any point on ℓ satisfy the inequalities $\mathcal{P}(\lambda_s) \geq 0$, $s = 1, \dots, d$, where

$$\mathcal{P}(x) = (a_1 - x) \dots (a_d - x)(\alpha_1 - x) \dots (\alpha_{d-1} - x).$$

Proof. Follows from [13]. □

Suppose that a bounded domain $\Omega \subset \mathbb{R}^d$ is given such that its boundary $\partial\Omega$ lies in the union of several quadrics from the family (1). Then, in elliptic coordinates, Ω is given by:

$$\beta'_1 \leq \lambda_1 \leq \beta''_1, \quad \dots, \quad \beta'_d \leq \lambda_d \leq \beta''_d,$$

where $a_{s+1} \leq \beta'_s < \beta''_s \leq a_s$ for $1 \leq s \leq d-1$ and $-\infty < \beta'_d < \beta''_d \leq a_d$.

Consider a billiard system within Ω and let $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ be caustics of one of its trajectories. For any $s = 1, \dots, d$, the set Λ_s of all values taken by the coordinate λ_s on the trajectory is, according to lemma 1, included in $\Lambda'_s = \{\lambda \in [\beta'_s, \beta''_s] : \mathcal{P}(\lambda) \geq 0\}$. By [14], the set Λ'_s is a closed interval and coincides with Λ_s . Denote $[\gamma'_s, \gamma''_s] := \Lambda_s = \Lambda'_s$.

Note that the trajectory touches quadrics of any pair $Q_{\gamma'_s}, Q_{\gamma''_s}$ alternately. Thus, any periodic trajectory has the same number of intersection points with each of them.

Theorem 1 A trajectory of the billiard system within Ω with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ is periodic with exactly n_s points at $Q_{\gamma'_s}$ and n_s points at $Q_{\gamma''_s}$ ($1 \leq s \leq d$) if and only if

$$\sum_{s=1}^d n_s (\mathcal{A}(P_{\gamma'_s}) - \mathcal{A}(P_{\gamma''_s})) = 0 \quad (2)$$

on the Jacobian of the curve

$$\Gamma : y^2 = \mathcal{P}(x) := (a_1 - x) \cdots (a_d - x)(\alpha_1 - x) \cdots (\alpha_{d-1} - x).$$

Here \mathcal{A} denotes the Abel-Jacobi map, where $P_{\gamma'_s}, P_{\gamma''_s}$ are points on Γ with coordinates $P_{\gamma'_s} = (\gamma'_s, (-1)^s \sqrt{\mathcal{P}(\gamma'_s)})$, $P_{\gamma''_s} = (\gamma''_s, (-1)^s \sqrt{\mathcal{P}(\gamma''_s)})$.

Proof. Following Jacobi [15], let us consider the integrals

$$\sum_{s=1}^d \int \frac{d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}, \quad \sum_{s=1}^d \int \frac{\lambda_s d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}, \quad \dots, \quad \sum_{s=1}^d \int \frac{\lambda_s^{d-1} d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}}$$

over the polygonal line $A_1 A_2 \dots A_{k+1}$, which represents a billiard trajectory, where $k = 2(n_1 + \dots + n_d)$. The last integral is equal to the total length of the polygonal line, while the others are equal to zero. Considering the behaviour of elliptic coordinates along each segment of the trajectory, we calculate values of the integrals and obtain that the condition $A_{k+1} = A_1$ is equivalent to (2). \square

Our next step is to introduce a notion of bounces ‘from outside’ and ‘from inside’. More precisely, let us consider an ellipsoid Q_λ from the confocal family (1) such that $\lambda \in (a_{s+1}, a_s)$ for some $s \in \{1, \dots, d\}$, where $a_{d+1} = -\infty$.

Observe that along a billiard ray which reflects at Q_λ , the elliptic coordinate λ_i has a local extremum at the point of reflection.

Definition 1 A ray reflects from outside at the quadric Q_λ if the reflection point is a local maximum of the Jacobian coordinate λ_s , and it reflects from inside if the reflection point is a local minimum of the coordinate λ_s .

Let us remark that in the case when Q_λ is an ellipsoid, the notions introduced in definition 1 coincide with the usual ones.

Assume now a k -tuple of confocal quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$ from the confocal pencil (1) is given. We consider a billiard system with trajectories having bounces at $Q_{\beta_1}, \dots, Q_{\beta_k}$ respectively. Such a trajectory has $d-1$ caustics from the same family (1). We additionally assign to each trajectory the signature $\sigma = (i_1, \dots, i_k)$ by the following rule:

$$\begin{aligned} i_s &= +1 && \text{if the reflection at } Q_{\beta_s} \text{ is from inside} \\ i_s &= -1 && \text{if the reflection at } Q_{\beta_s} \text{ is from outside.} \end{aligned}$$

Suppose $Q_{\beta_1}, \dots, Q_{\beta_k}$ are ellipsoids and consider a billiard ordered game with signature $\sigma = (i_1, \dots, i_k)$. In order that trajectories of such a game stay bounded, the following condition has to be satisfied:

$$i_s = -1 \Rightarrow i_{s+1} = i_{s-1} = 1 \quad \text{and} \quad \beta_{s+1} < \beta_s, \quad \beta_{s-1} < \beta_s.$$

(Here, we identify indices 0 and $k + 1$ with k and 1, respectively.)

Theorem 2 *Given a billiard ordered game within k ellipsoids $Q_{\beta_1}, \dots, Q_{\beta_k}$ with signature $\sigma = (i_1, \dots, i_k)$. Its trajectory with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$ is k -periodic if and only if*

$$\sum_{s=1}^k i_s (\mathcal{A}(P_{\beta_s}) - \mathcal{A}(P_\alpha))$$

is equal to a sum of several expressions of the form $\mathcal{A}(P_{\alpha_p}) - \mathcal{A}(P_{\alpha_{p'}})$ on the Jacobian of the curve $\Gamma : y^2 = \mathcal{P}(x)$, where $P_{\beta_s} = (\beta_s, +\sqrt{\mathcal{P}(\beta_s)})$, $\alpha = \min\{a_d, \alpha_1, \dots, \alpha_{d-1}\}$ and $Q_{\alpha_p}, Q_{\alpha_{p'}}$ are pairs of caustics of the same type.

When $Q_{\beta_1} = \dots = Q_{\beta_k}$ and $i_1 = \dots = i_k = 1$ we obtain the Cayley-type condition for billiard motion inside an ellipsoid in \mathbb{R}^d . Such periodic trajectories were described in [8] using a different technique, based on a Veselov-Moser discrete Lax representation.

We are going to treat in more detail the case of billiard motion between two ellipsoids.

Proposition 1 *The condition that there exists a closed billiard trajectory between two ellipsoids Q_{β_1} and Q_{β_2} , which bounces exactly m times to each of them, with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-1}}$, is*

$$\text{rank} \begin{pmatrix} f'_1(P_{\beta_2}) & f'_2(P_{\beta_2}) & \dots & f'_{m-d+1}(P_{\beta_2}) \\ f''_1(P_{\beta_2}) & f''_2(P_{\beta_2}) & \dots & f''_{m-d+1}(P_{\beta_2}) \\ \dots & \dots & \dots & \dots \\ f_1^{(m-1)}(P_{\beta_2}) & f_2^{(m-1)}(P_{\beta_2}) & \dots & f_{m-d+1}^{(m-1)}(P_{\beta_2}) \end{pmatrix} < m - d + 1.$$

Here

$$f_j = \frac{y - B_0 - B_1(x - \beta_1) - \dots - B_{d+j-2}(x - \beta_1)^{d+j-2}}{x^{d+j-1}} \quad 1 \leq j \leq m - d + 1$$

and $y = B_0 + B_1(x - \beta_1) + \dots$ is the Taylor expansion around the point symmetric to P_{β_1} with respect to the hyperelliptic involution of the curve Γ . (All notation is as in theorem 2.)

4 Periodic trajectories of billiards on quadrics in \mathbb{R}^d

In [16] the billiard systems on a quadric \mathcal{E} in \mathbb{R}^d

$$\frac{x_1^2}{a_1} + \dots + \frac{x_d^2}{a_d} = 1, \quad a_1 > \dots > a_d,$$

are defined as limits of corresponding billiards within \mathcal{E} , when one of the caustics tends to \mathcal{E} . The boundary of such a billiard consists of the intersection of \mathcal{E} with certain confocal quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$. The question of description of periodic trajectories of these systems was formulated as an open problem by Abenda and Fedorov [11].

By applying the limit procedure, from theorem 1, we obtain the following:

Theorem 3 *A trajectory of the billiard system constrained to the ellipsoid \mathcal{E} within $\Omega : \beta'_1 \leq \lambda_1 \leq \beta''_1, \dots, \beta'_{d-1} \leq \lambda_{d-1} \leq \beta''_{d-1}$, with caustics $Q_{\alpha_1}, \dots, Q_{\alpha_{d-2}}$, is periodic with exactly n_s bounces at each of quadrics $Q_{\gamma'_s}, Q_{\gamma''_s}$ ($1 \leq s \leq d-2$) if and only if*

$$\sum_{s=1}^{d-1} n_s (\bar{\mathcal{A}}(P_{\gamma'_s}) - \bar{\mathcal{A}}(P_{\gamma''_s})) = 0$$

on the Jacobian of the curve

$$\Gamma_1 : y^2 = \mathcal{P}_1(x) := -x(a_1 - x) \cdots (a_d - x)(\alpha_1 - x) \cdots (\alpha_{d-2} - x).$$

Here $P_{\gamma'_s}, P_{\gamma''_s}$ are the points on Γ_1 with coordinates $P_{\gamma'_s} = (\gamma'_s, (-1)^s \sqrt{\mathcal{P}_1(\gamma'_s)})$, $P_{\beta''_s} = (\gamma''_s, (-1)^s \sqrt{\mathcal{P}_1(\gamma''_s)})$, with $[\gamma'_s, \gamma''_s] = \{\lambda \in [\beta'_s, \beta''_s] : \mathcal{P}_1(\lambda) \geq 0\}$, $1 \leq s \leq d-2$, and $\bar{\mathcal{A}}(P) = (0, \int_0^P \frac{xdx}{y}, \int_0^P \frac{x^2dx}{y}, \dots, \int_0^P \frac{x^{d-2}dx}{y})$.

In the same way as in the previous section, a billiard ordered game constrained to the ellipsoid \mathcal{E} within given quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$ of the same type can be defined. The only difference is that now the signature $\sigma = (i_1, \dots, i_k)$ can be given arbitrarily, since trajectories are bounded, lying on the compact hypersurface \mathcal{E} . Denote by $Q_{\alpha_1}, \dots, Q_{\alpha_{d-2}}$ the caustics of a given trajectory of the game. Since quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$ are all of the same type, there exist μ', μ'' in the set $S = \{a_1, \dots, a_d, \alpha_1, \dots, \alpha_{d-2}\}$ such that $\beta_1, \dots, \beta_k \in [\mu', \mu'']$ and $(\mu', \mu'') \cap S$ is empty.

Associate with the game the following divisors on the curve Γ_1 :

$$\mathcal{D}_s = \begin{cases} P_{\mu''} & \text{if } i_s = i_{s+1} = 1 \\ 0 & \text{if } i_s = -i_{s+1} = 1, \beta_s < \beta_{s+1} \text{ or } i_s = -i_{s+1} = -1, \beta_s > \beta_{s+1} \\ P_{\mu''} - P_{\mu'} & \text{if } i_s = -i_{s+1} = 1, \beta_s > \beta_{s+1} \\ P_{\mu'} - P_{\mu''} & \text{if } i_s = -i_{s+1} = -1, \beta_s < \beta_{s+1} \\ P_{\mu'} & \text{if } i_s = i_{s+1} = -1, \end{cases}$$

where $P_{\mu'}$ and $P_{\mu''}$ are its branching points with coordinates $(\mu', 0)$ and $(\mu'', 0)$, respectively.

Theorem 4 *Given a billiard ordered game constrained to \mathcal{E} within quadrics $Q_{\beta_1}, \dots, Q_{\beta_k}$ with signature $\sigma = (i_1, \dots, i_k)$. Its trajectory with caustics $Q_{\alpha_1}, \dots,$*

$\dots, Q_{\alpha_{d-2}}$ is k -periodic if and only if

$$\sum_{s=1}^k i_s (\bar{\mathcal{A}}(P_{\beta_s}) - \bar{\mathcal{A}}(\mathcal{D}_s))$$

is equal to a sum of several expressions of the form $\bar{\mathcal{A}}(P_{\alpha_p}) - \bar{\mathcal{A}}(P_{\alpha_{p'}})$ on the Jacobian of the curve $\Gamma_1 : y^2 = \mathcal{P}_1(x)$, where $P_{\beta_s} = (\beta_s, +\sqrt{\mathcal{P}_1(\beta_s)})$ and $Q_{\alpha_p}, Q_{\alpha_{p'}}$ are pairs of caustics of the same type.

Proposition 2 Consider the case $d = 3$ and a billiard system constrained to the ellipsoid \mathcal{E} with the boundary Q_γ and caustic Q_α , $a_3 < \gamma < \alpha < a_2$. A trajectory is k -periodic if:

$$\text{rank} \begin{pmatrix} C_{p+1} & C_{p+2} & \dots & C_{2p-2} \\ C_{p+2} & C_{p+3} & \dots & C_{2p-1} \\ \dots & & & \\ C_{2p} & C_{2p+1} & \dots & C_{3p-3} \end{pmatrix} < p-2 \quad k = 2p$$

and

$$\text{rank} \begin{pmatrix} C_{p+1} & C_{p+2} & \dots & C_{2p-1} \\ C_{p+2} & C_{p+3} & \dots & C_{2p} \\ \dots & & & \\ C_{2p} & C_{2p+1} & \dots & C_{3p-2} \end{pmatrix} < p-1 \quad k = 2p+1$$

where

$$y = C_0 + C_1 \left(\tilde{x} - \frac{1}{\alpha - \gamma} \right) + C_2 \left(\tilde{x} - \frac{1}{\alpha - \gamma} \right)^2 + \dots$$

is the Taylor expansion with respect to $\tilde{x} = \frac{1}{\alpha - x}$ around the point P_γ .

5 Conclusion

As an important historical remark, we would like to emphasize the significance of Darboux's contribution to the study of problems related to generalized Poncelet theorem. The impression is that his work in the field (see [12]) is completely unknown nowadays. We shall present in another publication a more detailed overview of Darboux's ideas with comparison to the Lebesgue geometric approach and applications to separable perturbed problems [17, 18].

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